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# Orientation of Spinning Satellites by Radiation Pressure

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**By means of an array of mirrors or other optical devices, the force of radiation pressure is used to orient the spin axis of an artificial earth or sun satellite to the sun. The array also may produce spin-up or despin torques, be free of such torques, or regulate the spin rate to a predetermined value. This method of orientation, being stable, passive, and heliotropic, is particularly advantageous for small artificial satellites that derive their electrical power from solar cells.**

## Nomenclature

$A$ = area of plane surface intercepting radiation
$A_s$ = area of spin regulating surface
$F_a$ = force parallel to spin axis
$F_N$ = force normal to surface
$F_s$ = force perpendicular to spin axis
$F_T$ = force tangent to surface
$i$ = angle of incidence of radiation from normal to surface
$I$ = major principal moment of inertia
$p_s$ = solar-radiation pressure
$R$ = mean distance of a surface from the spin axis
$R_i$ = inner radius of annular array
$R_o$ = outer radius of annular array
$R_s$ = mean distance of spin regulating surface from the spin axis
$t$ = time, sec
$T$ = torque
$T_p$ = torque perpendicular to spin axis
$T_s$ = despin torque
$\alpha$ = angle between reflector surface and spin axis
$\phi$ = angle about spin axis
$\theta$ = tracking error angle between angular momentum vector and radiation vector
$\omega$ = spin rate, rad/sec

## Introduction

The use of solar-radiation pressure for attitude control of an artificial satellite that will be described differs from previous methods in that it can be used only for a spinning vehicle. Methods of orientation by solar-radiation pressure suitable for nonspinning vehicles have been described by Sohn,[1] Frye and Stearns,[2] Newton,[3] and Villers and Olha.[4]

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The method consists of an array of mirrors or other optical devices on the spinning satellite, arranged to acquire angular momentum from radiation pressure that will make the spin axis precess to the source of radiation.

## Required Attitude Control Torque

To be continually directed to the sun, the spin axis of an earth satellite must be precessed at the rate of only $2 \times 10^{-7}$ rad/sec. Accordingly, the torque required for precession, being very small, is compatible with the forces available from radiation pressure. An earth satellite with a major principal moment of inertia of 10 kg m$^2$ turning at 1 rpm would require only 0.02 dyne-m of torque to be developed from a solar-radiation pressure of about 1 dyne/m$^2$.

## Mechanisms for Producing Precession Torques from Radiation Pressure

Radiation pressure cannot be made to produce a pure torque; however, forces due to light pressure on a rigid body can be produced such that in addition to a net push, there will be unbalanced forces about the center of inertia. Such torques have been the means proposed for orienting nonspinning satellites.[1–4]

The design of devices to produce a torque of fixed direction on a spinning satellite is somewhat more involved. The forces on different parts of the satellite cannot be made to differ simply, as any net effect on angular momentum in 1 revolution would be zero. Rather, the force of light pressure should be made dependent on the angle of incidence on a particular surface element in such a manner that, as the satellite spins, angular momentum is accumulated.

The torque that is required is not such as to spin-up or despin the satellite but is in a direction perpendicular to the spin axis. Such a torque may be caused to precess the spin axis in the desired direction.

Any arrangements of mirrors and other devices that can be devised to produce the desired precession torque must be examined for their effect on spin rate. The following effects are possible:

1) A despin torque may be produced which will eventually reverse the direction of rotation, resulting in deorientation.

2) A despin torque of second order may be produced which will cause the spin rate (and hence the tracking error) to approach zero asymptotically as an inverse function of time.

3) The spin rate may be unaffected.

4) A spin-up torque for low rates of spin and a despin torque for high rates of spin may provide automatic regulation of spin rate.

5) A second-order spin-up torque may be produced which would increase the spin rate (and hence the tracking error), resulting eventually in deorientation.

These possibilities are examined for two symmetrical vehicle configurations: the windmill array and the corner mirror array.

## Windmill Array

The windmill array is an example of the first type and consists of an annular array of radial flat reflector blades set at an angle $\alpha$ from the spin axis. A pair of such blades of area $A$ on a common diameter produces both despin and precession torques, which may be derived from Fig. 1.

The normal force on one blade is

$$F_N = 2Ap_s \cos^2 i = 2Ap_s \sin^2(\alpha + \theta) \quad (1)$$

where $p_s$ is the solar radiation pressure on an absorbing body and $\theta$ is the tracking error, assuming that the common diameter at this particular instant is perpendicular to the rays of the sun. The axial component of force $F_a$ on one blade is

$$F_a = 2Ap_s \sin^2(\alpha + \theta) \sin\alpha \quad (2)$$

and the side component $F_s$ is

$$F_s = 2Ap_s \sin^2(\alpha + \theta) \cos\alpha \quad (3)$$

The torque $T_p$ producing the desired precession is equal to the difference between the axial forces on a pair of diametrically opposed blades times the mean radius $R$ or

$$T_p = 2RAp_s[\sin^2(\alpha + \theta) - \sin^2(\alpha - \theta)] \sin\alpha = 2RAp_s \sin\alpha \sin 2\alpha \sin 2\theta \quad (4)$$

The despin torque $T_s$ is equal to the sum of the side forces on a pair of diametrically opposed blades times the mean radius $R$ or

$$T_s = 2RAp_s[\sin^2(\alpha + \theta) + \sin^2(\alpha - \theta)] \cos\alpha = 2RAp_s[1 - \cos 2\alpha \cos 2\theta] \cos\alpha \quad (5)$$

which is not zero even when $\theta$, the tracking error, is zero. Thus, this despin torque effectively would reverse the direction of rotation of the satellite, reverse also the direction of precession, and cause deorientation. The windmill array may be said to be heliophobic and may be useful where automatic deorientation from the sun is desired.

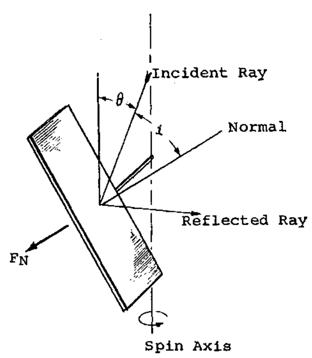


**Fig. 1 Force on one mirror of the windmill array**

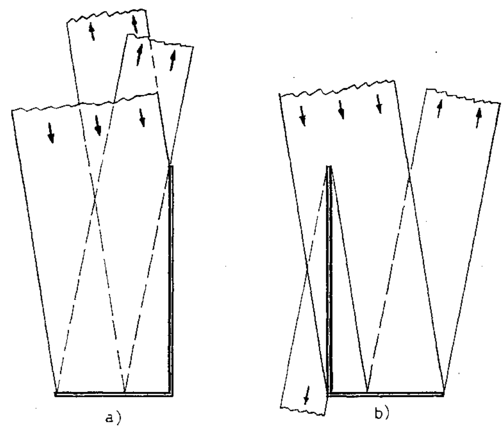


**Fig. 2 Incident and reflected light beams on a pair of corner mirrors**

## Corner Mirror Array

The corner mirror array is truly heliotropic, being an example of the second type of array having a second-order despin torque that results in a gradual lessening of the tracking error. These mirrors also may be arranged radially in an annular array about the spin axis. One such corner mirror, having a height twice its base, is shown in Fig. 2a, with light incident on the inside of the corner at an angle $\theta$ with respect to the long side that is parallel to the spin axis. The sum of the forces $F_a$ parallel to the spin axis produced by light pressure for this configuration is

$$F_a = 2Ap_s \cos\theta(\cos\theta + 2\sin\theta) \quad (6)$$

while the side force $F_s$ is

$$F_s = 8Ap_s \sin^2\theta \quad (7)$$

where $A$ is the area of the base of the corner mirror.

For a pair of such corner mirrors on a diameter normal to the sun's rays, one corner mirror will be illumined as shown in Fig. 2a, and the other will have light incident on the outside of the corner at an angle $\theta$ as shown in Fig. 2b. The outside of the corner is assumed to have a mirror surface, in which case the axial force $F_a$ is given by Eq. (6) (with $\theta$ negative), and the side force is one-half that given by Eq. (7). The precession torque $T_p$, being the product of the radius $R$ and the difference in axial forces, is

$$T_p = 2RAp_s \cos\theta[\cos\theta + 2\sin\theta - \cos\theta + 2\sin\theta] = 4RAp_s \sin 2\theta \quad (8)$$

The despin torque $T_s$, given by the product of the radius and the difference in the side forces, is

$$T_s = 4RAp_s \sin^2\theta \quad (9)$$

Since the despin torque rapidly approaches zero as $\theta$ approaches zero, it may be made far less than the precession torque. However, even this second-order despin torque can be avoided by extending the long side of the mirror an equal distance past the short side as shown in Fig. 3. When light shines on the inside corner, the extended portion is in shadow and the forces given by Eqs. (6) and (7) are as before. For the other member of a pair of such mirrors, light shines on the extended outside surface, with no increase in the axial force, Eq. (6), in spite of the larger surface, since light pressure will not produce forces parallel to a mirror surface. However, because of the larger area of this outside surface, the side force on it is doubled, equaling that on the inside surface of the other corner mirror of the pair. Thus, the net

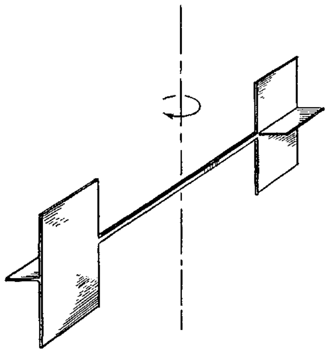

**Fig. 3 Pair of double-corner mirrors having zero despin torque**

despin torque produced by such a pair of double-corner mirrors is zero.

## Tracking Error of the Corner Mirror Array

To find the precession torque produced by a spinning annular array of corner mirrors, the torque produced by one pair may be averaged over 1 revolution. The torque, however, varies as the absolute value of the cosine of the angle $\phi$ by which the mutual diameter of a pair of corner mirrors departs from the position where it is normal to the sun's rays. The average torque for one pair of mirrors will, therefore, be 63% ($2/\pi$) of that given by Eq. (8). Since, however, in an annular array, the corner mirrors will be wedge-shaped, it is more accurate to integrate Eq. (8), using both the radius and angle as a variable. To do so, the cosine of the angle $\phi$ must be taken into account as well as the empty spaces between adjacent mirrors, which must be equal to the base width of one mirror to prevent mutual shadowing. The incremental torque $dT_p$ produced by a pair of infinitesimal corner mirrors to be integrated is then

$$dT_p = (2p_s \sin 2\theta)R^2|\cos\phi|dR\, d\phi \qquad (10)$$

whence

$$T_p = (\tfrac{4}{3})p_s \sin 2\theta(R_o^3 - R_i^3) \qquad (11)$$

where $R_o$ and $R_i$ are the outer and inner radii of the annular array, respectively. A photograph of an array of such mirrors designed to orient a solar panel is shown in Fig. 4. For this configuration the tracking error angle $\theta$ is given by

$$\sin 2\theta = \frac{1.5 \times \omega I \times 10^{-7}}{p_s(R_o^3 - R_i^3)} \qquad (12)$$

in terms of satellite angular velocity $\omega$ and major moment of inertia $I$. For the typical values $R_o = 1$ m, $R_i = 0.75$ m, $I = 10$ kg m², $p_s = 0.466$ dynes/$m^2$, the tracking error is 1.6° at 1 rpm. For earth satellites that pass through the earth's shadow once each orbit, the tracking error can be estimated by assuming a corresponding lower value of light pressure $p_s$.

## Spin Rate Control by Light Pressure

Where it is desired to hold the spin rate to the optimum value, one not so small as to be upset by disturbances or so high as to entail large tracking errors or require large arrays, the array may be supplemented by devices that tend to cancel and reverse the despin torque when the spin rate (and, therefore, the tracking error) falls below a predetermined value. One such device is a flat surface, one side of which is absorbent (black) and the other side diffusely reflecting (white). These surfaces may be considered in pairs on a common diameter of $2R_s$ about the spin axis. On one member of such a pair, aligned with the spin axis, light will fall at an angle of incidence

$$i = \pi/2 - \theta \qquad (13)$$

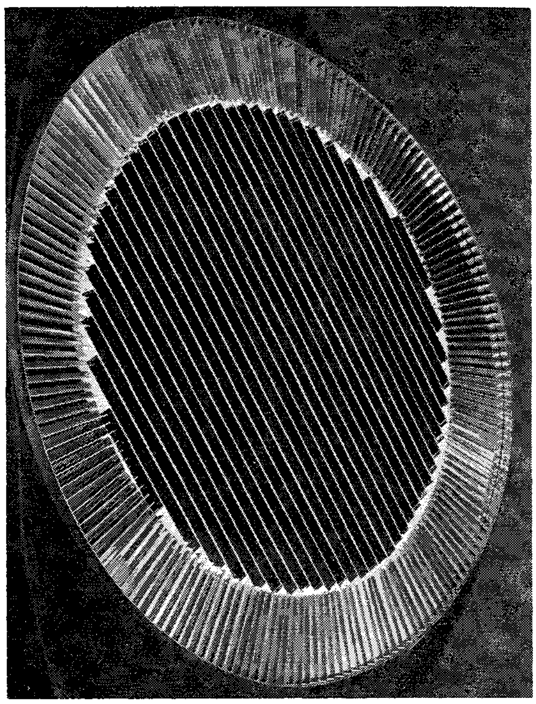

**Fig. 4 Model of solar cell panel with self-orienting mirror array**

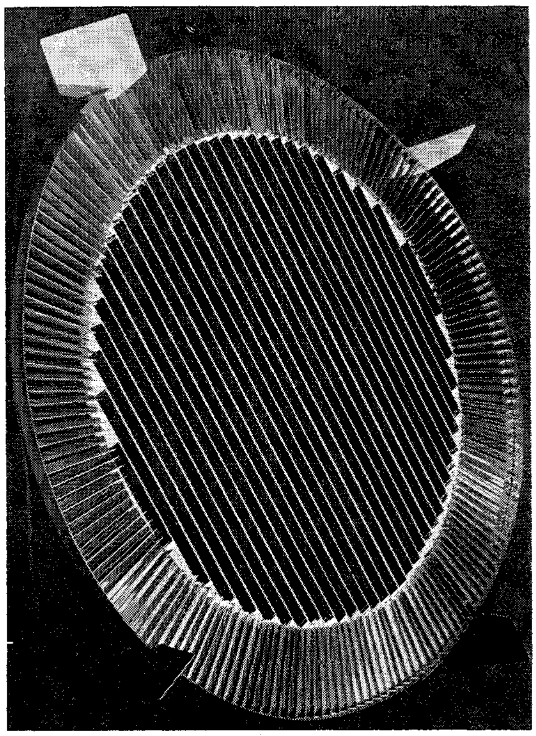

**Fig. 5 Solar cell panel and self-orienting mirror array with spin stabilizing surfaces attached**

on the black side, $\theta$ being the tracking error angle with respect to the spin axis. The axial and side forces will be, respectively,

$$F_a = A_s p_s \sin\theta \cos\theta \qquad (14)$$

$$F_s = A_s p_s \sin^2\theta \qquad (15)$$

where $A_s$ is the area of the surface. The forces produced by light falling on the white side of the other member of the pair will be

$$F_a = A_s p_s \sin\theta \cos\theta \qquad (16)$$

$$F_s = A_s p_s \sin\theta[\sin\theta + \tfrac{2}{3}] \quad (17)$$

taking into account the diffusely reflecting surface. The resulting precession torque $T_p$ is zero since the axial forces given by Eq. (14) and Eq. (16) are equal; however, the spin torque (which will be used to oppose despin) is equal to

$$T_s = \tfrac{2}{3} A_s p_s R_s \sin\theta \quad (18)$$

provided care is taken to make the $IR$ emissivities of both black and white sides equal.

Black and white surfaces of the forementioned type may be used with corner mirrors, such as shown in Fig. 2, without affecting the precession torque of the latter but in a manner such as to oppose the despin torque. If this is done, the expression for the despin torque of the combination is the difference between Eqs. (9) and (18) or

$$T_s = p_s \sin\theta[4RA \sin\theta - \tfrac{2}{3} A_s R_s] \quad (19)$$

For

$$\sin\theta = \tfrac{1}{6}[A_s R_s / AR] \quad (20)$$

there will be neither a spin-up nor a despin torque, but a spin-up torque for smaller values of $\theta$ (when the spin rate is also small) and a despin torque for larger values of $\theta$ (when the spin rate is higher). Equation (20), together with Eq. (12), gives the asymptotic values of both angular momentum and tracking error, although the asymptotic tracking error is determined by Eq. (20) alone. The smaller the relative size of the spin regulating array, the smaller this tracking error. The array of black and white surfaces may be radial extensions of the long side of the corner mirrors but, if the latter are close together, mutual interference may be avoided by using a smaller number of larger surfaces, one surface being sufficient. Control of the asymptotic spin rate is also possible by adjusting the angle of attack of the spin regulating surfaces by remote control or by centrifugal force. A photograph of an attitude stabilizing array of corner mirrors having spin regulating surfaces is shown in Fig. 5.

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# Singular Extremals in Lawden's Problem of Optimal Rocket Flight

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**The problem of optimal rocket flight in an inverse square law force field has been studied extensively by Lawden and Leitmann. Periods of zero thrust, intermediate thrust, and maximum thrust are possible subarcs of the solution according to analysis of the Euler-Lagrange equations and the Weierstrass necessary condition. Arcs of intermediate thrust have been examined recently by Lawden; however, the question of whether or not such arcs actually may furnish a minimum has been left unresolved. The present paper derives the singular extremals of Lawden's problem by means of the Legendre-Clebsch necessary condition applied in a transformed system of state and control variables. These are obtained as circular orbits along which the thrust is zero and intermediate thrust arcs are found in Lawden's analysis. Since these solutions satisfy only the weak form of the Legendre-Clebsch condition, i.e., the extremals are singular in the transformed system of variables, the question of their minimality remains unanswered.**

## Introduction

The problem of optimal rocket flight in an inverse square law force field has been investigated by Lawden[1, 2] and Leitmann.[3] Although considerable progress has been made in the study of properties of the solution, a question remains as to the possible appearance of subarcs of intermediate thrust.[4, 5] Such arcs are among the *singular* extremals of the problem, in classical variational terminology, and are resistant to analytical efforts owing to the unavailability of a general theory applicable to singular cases.[6]

This paper first presents a brief development of the Euler-Lagrange equations and the Weierstrass necessary condition along the lines of previous investigations and then proceeds to an analysis of intermediate thrust arcs.

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## Lawden's Problem

The equations of motion for a rocket in two-dimensional flight are given by

$$\dot{u} = (T/m) \sin\theta + Y \quad (1)$$